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Derivation of the Mathematical Transfer Function of an Electrodynamic Vibration Exciter

C. P. Chapman

E. L. Sheldon

E. L. Sheldon, *Manager*
Environmental and Dynamic Testing Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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CONTENTS

FIGURES

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ABSTRACT

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The transfer function of an electrodynamic vibration system (power amplifier and vibration exciter) is defined as the ratio of table accelerations to input voltage. The equation is itself a function of frequency. An input signal with a given frequency is applied to the input of the system; the signal appears at the output, multiplied by the magnitude of the transfer function, with its phase shifted by the phase angle of the transfer function.

This Report presents a detailed solution of the mathematical equation that describes a typical electrodynamic vibration system. The mathematical analysis is compared with the empirical transfer function of a particular vibration exciter to demonstrate the validity of the equation and the value of mathematics and computers for analysis of vibration techniques.

I. INTRODUCTION

Transfer functions of a vibration system can be obtained empirically by plotting the voltage required by the system to maintain a certain (constant) acceleration level. It is a more difficult task to obtain the transfer function by mathematical methods. A mathematical transfer function of a vibration system is a desirable equation and valuable tool for investigating the dynamics of these systems, and in particular, the equation is important in the analysis, synthesis, or modification of control techniques for single or multiple shaker systems.

Although mathematical transfer functions of vibration systems have appeared in the literature from time to time, there have been two problems associated with these equations: (1) there have been errors in the equations in some cases, and (2) the derivations have not been included, or if included, are not obvious or not complete.

II. THE ANALOG CIRCUIT

The equivalent electrical circuit for an electrodynamic shaker (Fig. 1) is used quite often for analysis and synthesis of vibration systems[1] (Ref. 1). In this figure,

v_d = Driver coil voltage, volts

i_d = Driver coil current, amp

R_d = Driver coil resistance, ohms

L_d = Driver coil inductance, henrys

K_g = Force constant, newtons/amp
(The ideal transformer with turns ratio K_g:1 provides the electromechanical conversion.)

M_c = Mass of the driver coil, kg

M_t = Table mass, kg

$1/K_t$ = Table compliance, m/newton

$1/K_f$ = Flexure compliance, m/newton

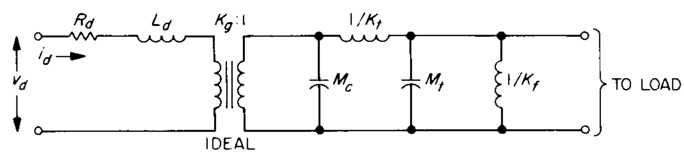

Fig. 1. Equivalent electrical analog circuit for an electrodynamic shaker

The following relationships will transform the above mechanical terms to analog electrical terms[2]:

$1/K \rightarrow L$: inductance, henrys

$M \rightarrow C$: capacitance, farads

$F \rightarrow I$: current, amp

$\dot{x} \rightarrow V$: voltage, volts

$$\ddot{x} \rightarrow \frac{d}{dt} V = SV: \text{ time derivative of voltage}$$

The subscripts from the mechanical system are carried over to the electrical analog. Also, $S = d/dt$ and $1/S = \int dt$, where S is the Laplace transform variable.

Since the equivalent electrical circuit has an ideal transformer with turns ratio K_g:1, it is necessary to refer the secondary side of the transformer over to the primary side. This can be done in the following manner (Ref. 3): (1) multiply

[1]Ling Electronics Co. and MB Electronics Co.—two prominent manufacturers of vibration equipment—have used this equivalent circuit in their literature.

[2]The force–current analogy system is employed because from the point of view of physical interpretation, it is the only natural analogy (Ref. 2).

inductances by K_g^2 and (2) multiply capacitances by $1/K_g^2$. The resulting equivalent electrical circuit is shown in Fig. 2.

The voltage at node 2 is $V_{2L_f} = SK_g^2 L_f I_3$.

The voltage transfer function of the circuit (Fig. 2) is V_{2L_f}/v_d. To obtain this transfer function, it is necessary to calculate I_3.

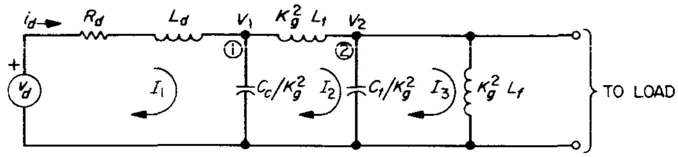

Fig. 2. Reflected equivalent electrical analog circuit

For the analog system, voltage is equal to velocity. The velocity at node 2 is $\dot{x}$. The acceleration at node 2 is $\ddot{x}$, where x is the displacement. Therefore, $dx/dt = Sx$ is the velocity and $d^2x/dt^2 = S^2x$ is the acceleration. Again, the velocity at node 2 (the voltage of the analog system) is $SK_g^2 L_f I_3$. The acceleration at node 2 is $S^2 K_g^2 L_f I_3$. The voltage transfer function in terms of acceleration is defined as

$$\frac{\ddot{x}_2}{v_d} \text{ or } \frac{S\dot{x}_2}{v_d} \text{ or } \frac{SV_2}{v_d}$$

III. MODIFICATION OF THE ANALOG CIRCUIT

Physically, however, one is not too interested in the acceleration of the flexures, but rather in the acceleration of the vibration table. Although the acceleration at node 2 is the same as that for the flexures, the forces (currents in the analog system) are different:

$$V_{2C_t} = \frac{1}{S(C_t/K_g^2)} (I_2 - I_3)$$

For the physical interpretation of the analog circuit, it may be better to redraw the circuit as shown in Fig. 3.

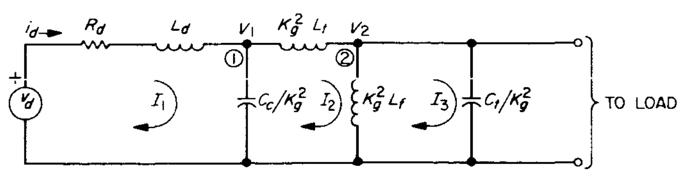

Fig. 3. Modified equivalent electrical analog circuit

IV. DERIVATION OF THE TRANSFER FUNCTION

In the new circuit, the velocity of the vibration table is

$$V_{2c_t} = \frac{1}{S(C_t/K_g^2)} I_3$$

To obtain the acceleration of the table, it is only necessary to multiply the above equation by the Laplace operator S

$$S\, V_{2c_t} = \frac{S}{S(C_t/K_g^2)} I_3 = \frac{K_g^2 I_3}{C_t}$$

The table acceleration is

$$\ddot{x}_2 = \frac{K_g^2 I_3}{C_t} \tag{1}$$

where I_3 is as defined in Fig. 3. The ratio $\ddot{x}/v_d$ is needed to obtain the mathematical transfer function.

By writing Kirchhoff's voltage equations (loop equations), a matrix $[z]$ can be constructed:

$$\begin{bmatrix} (R_d + SL_d + K_g^2/SC_c) & -K_g^2/SC_c & 0 \\ -\dfrac{K_g^2}{SC_c} & \left(\dfrac{K_g^2}{SC_c} + SK_g^2L_t + SK_g^2L_f\right) & -SK_g^2L_f \\ 0 & -SK_g^2L_f & \left(SK_g^2L_f + \dfrac{K_g^2}{SC_t}\right) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} v_d \\ 0 \\ 0 \end{bmatrix}$$

$$I_3 = \frac{\begin{vmatrix} \left(R_d + SL_d + \dfrac{K_g^2}{SC_c}\right) & -\dfrac{K_g^2}{SC_c} & v_d \\ -\dfrac{K_g^2}{SC_c} & \left(\dfrac{K_g^2}{SC_c} + SK_g^2L_t + SK_g^2L_f\right) & 0 \\ 0 & -SK_g^2L_f & 0 \end{vmatrix}}{\Delta_z}$$

where Δ_z is the determinant of the $[z]$ matrix.

The numerator of I_3 can be solved by expanding around v_d since the third column has 2 zeroes:

$$I_{3\ \text{numerator}} = v_d \left(\frac{K_g^2}{SC_c} \cdot \frac{SK_g^2 L_f}{1} \right)$$

$$= v_d (K_g^4 L_f / C_c)$$

Therefore

$$I_3 = \frac{(K_g^4 L_f / C_c) v_d}{\Delta_z} \tag{2}$$

By substituting Eq. (2) into Eq. (1), the table acceleration becomes

$$\ddot{x}_2 = \frac{v_d (K_g^4 L_f)/C_c}{\Delta_z} \left(\frac{K_g^2}{C_t} \right)$$

By dividing both sides of the above equation by v_d, the ratio $\ddot{x}_2/v_d$ is obtained, which is the transfer function of an electrodynamic vibration system:

$$\frac{\ddot{x}_2}{v_d} = \frac{K_g^6 L_f / C_c C_t}{\Delta_z} \tag{3}$$

Consider Δ_z:

$$\Delta_z = \begin{vmatrix} (R_d + SL_d + K_g^2/SC_c) & -K_g^2/SC_c & 0 \\ -K_g^2/SC_c & \left(\dfrac{K_g^2}{SC_c} + SK_g^2 L_t + SK_g^2 L_f \right) & -SK_g^2 L_f \\ 0 & -SK_g^2 L_f & (SK_g^2 L_f + K_g^2/SC_t) \end{vmatrix}$$

Expanding Δ_z around the third row:

$$\Delta_z = SK_g^2 L_f \begin{vmatrix} R_d + SL_d + K_g^2/SC_c & 0 \\ -K_g^2/SC_c & -SK_g^2 L_f \end{vmatrix} + (SK_g^2 L_f + K_g^2/SC_t)$$

$$\times \begin{vmatrix} (R_d + SL_d + K_g^2/SC_c) & -K_g^2/SC_c \\ -K_g^2/SC_c & \left(\dfrac{K_g^2}{SC_c} + SK_g^2 L_t + SK_g^2 L_f \right) \end{vmatrix}$$

$$\Delta_z = SK_g^2L_f\ [-SK_g^2L_f(R_d + SL_d + K_g^2/SC_c)] + (SK_g^2L_f + K_g^2/SC_t)$$

$$\times \left\{ \left[(R_d + SL_d + K_g^2/SC_c) \cdot \left(\frac{K_g^2}{SC_c} + SK_g^2L_t + SK_g^2L_f \right) \right] - (K_g^2/SC_c)^2 \right\}$$

Multiplying through,

$$\Delta_z = SK_g^2L_f[-SK_g^2L_fR_d - S^2K_g^2L_fL_d - SK_g^4L_f/SC_c] + (SK_g^2L_f + K_g^2/SC_t)$$

$$\times \left[\frac{R_dK_g^2}{SC_c} + R_dSK_g^2L_t + R_dSK_g^2L_f + \frac{SL_dK_g^2}{SC_c} + S^2K_g^2L_dL_t + S^2K_g^2L_dL_f \right.$$

$$\left. + \cancel{\left(\frac{K_g^2}{SC_c} \right)^2} + \frac{SK_g^4L_t}{SC_c} + \frac{SK_g^4L_f}{SC_c} - \cancel{\left(\frac{K_g^2}{SC_c} \right)^2} \right]$$

Continuing the multiplication,

$$\Delta_z = -S^2K_g^4L_f^2R_d - S^3K_g^4L_f^2L_d - \frac{S^2K_g^6L_f^2}{SC_c} + \frac{SK_g^4L_fR_d}{SC_c} + S^2K_g^4L_fR_dL_t$$

$$+ S^2K_g^4L_f^2R_d + \frac{S^2K_g^4L_fL_d}{SC_c} + S^3K_g^4L_fL_dL_t + S^3K_g^4L_f^2L_d + \frac{S^2K_g^6L_fL_t}{SC_c}$$

$$+ \frac{S^2K_g^6L_f^2}{SC_c} + \frac{R_dK_g^4}{S^2C_cC_t} + \frac{R_dSK_g^4L_t}{SC_t} + \frac{R_dSK_g^4L_f}{SC_t} + \frac{SL_dK_g^4}{S^2C_cC_t}$$

$$+ \frac{S^2K_g^4L_dL_t}{SC_t} + \frac{S^2K_g^4L_dL_f}{SC_t} + \frac{SK_g^6L_t}{S^2C_cC_t} + \frac{SK_g^6L_f}{S^2C_cC_t}$$

Canceling S's and like terms,

$$\Delta_z = \frac{K_g^4L_fR_d}{C_c} + S^2K_g^4L_fR_dL_t + \frac{SK_g^4L_fL_d}{C_c} + S^3K_g^4L_fL_dL_t + \frac{SK_g^6L_fL_t}{C_c}$$

$$+ \frac{R_dK_g^4}{S^2C_cC_t} + \frac{R_dK_g^4L_t}{C_t} + \frac{R_dK_g^4L_f}{C_t} + \frac{L_dK_g^4}{SC_cC_t} + \frac{SK_g^4L_dL_t}{C_t} + \frac{SK_g^4L_dL_f}{C_t}$$

$$+ \frac{K_g^6L_t}{SC_cC_t} + \frac{K_g^6L_f}{SC_cC_t}$$

The next step is to put Δ_z over a common denominator and collect terms with reference to the power of S:

By inspection, $\Delta_{z\ \text{denominator}} = S^2C_cC_t$

$$\Delta_{z\ \text{numerator}} = S^2C_tK_g^4L_fR_d + S^4K_g^4L_fR_dL_tC_cC_t + S^3K_g^4L_fL_dC_t + S^5K_g^4L_fL_dL_tC_cC_t + S^3K_g^6L_fL_tC_t + R_dK_g^4 + S^2R_dK_g^4L_tC_c + S^2R_dK_g^4L_fC_c + SL_dK_g^4 + S^3K_g^4L_dL_tC_c + S^3K_g^4L_dL_fC_c + SK_g^6L_t + SK_g^6L_f$$

$$\Delta_{z\ \text{numerator}} = S^5K_g^4L_fL_dL_tC_cC_t + S^4K_g^4L_fR_dL_tC_cC_t + [S^3K_g^4L_fL_dC_t + S^3K_g^6L_fL_tC_t + S^3K_g^4L_dL_tC_c + S^3K_g^4L_dL_fC_c] + [S^2C_tK_g^4L_fR_d + S^2R_dK_g^4L_tC_c + S^2R_dK_g^4L_fC_c] + [SL_dK_g^4 + SK_g^6L_t + SK_g^6L_f] + R_dK_g^4$$

It is convenient to have the first term of any polynomial by itself, i.e., in this case, $S^5 + S^4K_1 + S^3K_2 + \cdots + K_5$; therefore, by dividing both numerator and denominator of Δ_z by $K_g^4L_fL_dL_tC_cC_t$, the following equation is obtained:

$$\Delta_{z\ \text{numerator}} = S^5 + \frac{S^4R_d}{L_d} + \left(\frac{S^3}{L_tC_c} + \frac{S^3K_g^2}{L_dC_c} + \frac{S^3}{L_fC_t} + \frac{S^3}{L_tC_t}\right) + \left(\frac{S^2R_d}{L_dL_tC_c} + \frac{S^2R_d}{L_dL_fC_t} + \frac{S^2R_d}{L_dL_tC_t}\right) + \left(\frac{S}{L_fL_tC_cC_t} + \frac{SK_g^2}{L_dL_fC_cC_t} + \frac{SK_g^2}{L_dL_tC_cC_t}\right) + \frac{R_d}{L_fL_dL_tC_cC_t}$$

Collecting terms,

$$\Delta_{z\ \text{numerator}} = S^5 + S^4\left(\frac{R_d}{L_d}\right) + S^3\left(\frac{1}{L_tC_c} + \frac{K_g^2}{L_dC_c} + \frac{1}{L_fC_t} + \frac{1}{L_tC_t}\right) + S^2\frac{R_d}{L_d}\left(\frac{1}{L_tC_c} + \frac{1}{L_fC_t} + \frac{1}{L_tC_t}\right) + S\frac{1}{C_cC_t}\left(\frac{1}{L_fL_t} + \frac{K_g^2}{L_dL_f} + \frac{K_g^2}{L_dL_t}\right) + \frac{R_d}{L_fL_dL_tC_cC_t}$$

Dividing $\Delta_{z\ \text{denominator}}$ by $K_g^4L_fL_dL_tC_cC_t$, it becomes:

$$\Delta_{z\ \text{denominator}} = \frac{S^2}{K_g^4L_fL_dL_t}$$

The acceleration of the table is

$$\ddot{x}_2 = \frac{K_g^2 I_3}{C_t} \tag{4}$$

The force of the table is

$$I_3 = \frac{(K_g^4 L_f / C_c) v_d}{\Delta_z} \tag{5}$$

The desired transfer function is

$$\frac{\ddot{x}_2}{v_d} = \frac{K_g^6 L_f / C_c C_t}{\Delta_z} \tag{6}$$

if Δ_z is defined

$$\Delta_z \equiv \frac{P}{S^2 / K_g^4 L_f L_d L_t}$$

where P is the fifth-order polynomial ($\Delta_{z\ \text{numerator}}$). Then substituting Δ_z into Eq. (6) gives

$$\frac{\ddot{x}}{v_d} = \frac{K_g^6 L_f}{C_t C_c} \cdot \frac{1}{P/(S^2/K_g^4 L_f L_d L_t)}$$

or

$$\frac{\ddot{x}_2}{v_d} = \frac{K_g^6 L_f}{C_t C_c} \cdot \frac{S^2}{P(K_g^4 L_f L_d L_t)}$$

this becomes

$$\frac{\ddot{x}_2}{v_d} = \frac{S^2 K_g^2}{(C_t C_c L_d L_t) P}$$

so that

$$\frac{\ddot{x}_2}{v_d} = \frac{S^2 K_g^2 / C_t C_c L_d L_t}{P}$$

where, again, P is the fifth-order polynomial which is the numerator of Δ_z, and $\ddot{x}_2/v_d$ is the mathematical voltage transfer function of the electrodynamic shaker system that has an equivalent electrical analog circuit such as that of Fig. 1.

For the mechanical engineer who prefers mechanical terms rather than electrical terms to describe a transfer function, the function can be transformed back using the analogy in reverse, obtaining:

$$\ddot{x}/v_d = \frac{S^2 K_g^2 K_t / M_c M_t L_d}{S^5 + S^4\left(\frac{R_d}{L_d}\right) + S^3\left(\frac{K_t}{M_c} + \frac{K_g^2}{L_d M_c} + \frac{K_f}{M_t} + \frac{K_t}{M_t}\right) + S^2 \frac{R_d}{L_d}\left(\frac{K_t}{M_c} + \frac{K_f}{M_t} + \frac{K_t}{M_t}\right) + S \frac{1}{M_c M_t}\left(K_f K_t + \frac{K_g^2 K_f}{L_d} + \frac{K_g^2 K_t}{L_d}\right) + \frac{R_d K_f K_t}{M_c M_t L_d}}$$

V. COMPARISON WITH EMPIRICAL DATA

Transfer functions are frequently expressed in terms of the Laplace transform variable S (Ref. 4), as above. For steady state sinusoidal conditions, the Laplace variable S can be replaced by the operator $j\omega$, where $\omega = 2\pi F$ and $j = \sqrt{-1}$.

A computer program was written (Ref. 5) and an IBM 7090 was utilized to test the mathematical model of the vibration system, using the shaker constants of an MB Electronics C-10E exciter. Table 1 lists the constants specified by the manufacturer.

Table 1. Shaker constants of MB Electronics C-10E Exciter

Constant	Value	Mks Units
K_t	8.16×10^6	newtons/m
K_f	6.30×10^5	newtons/m
M_c	1.815	kg
M_t	6.120	kg
L_d	1.2×10^{-3}	henrys
R_d	3.0	ohms
K_g	190.0	newtons/amp

Figure 4 illustrates the results obtained by comparing the acceleration-vs-voltage plot obtained from the MB C-10E exciter (bare table) with that predicted by the computer with the analytically derived transfer function.

The results suggest, among other things, that perhaps the digital computer could be utilized to design—or modify—electrodynamic vibration exciters. It should be a very simple procedure to optimize a transfer function by computer

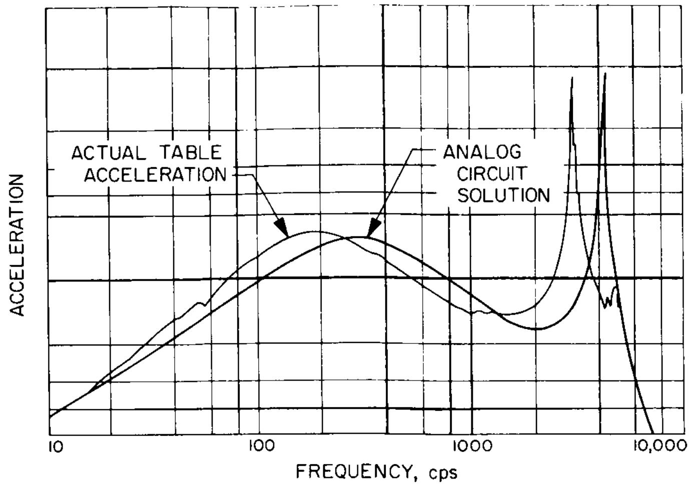

Fig. 4. Empirical transfer function vs mathematical transfer function of an electrodynamic vibration system

techniques. The computer program can be quite simple (Ref. 6) and the computer time to do this should not run over 4 min at the very most. All this presupposes an accurate mathematical model. Nothing is gained by having three models: one for the low frequencies, one for the midrange, and another for the high frequencies. In fact, *the* mathematical transfer function will suffice for all frequencies.

VI. CONCLUSIONS

The close correlation between the mathematical transfer function and the empirical transfer function suggests the validity of the mathematics and the analog circuit and the value of using mathematical and computer techniques to analyze and synthesize vibrations systems. It is important, however, that the model and mathematical equations be exact in order to produce meaningful results.

By extending the analog circuit and utilizing the techniques of this paper, it should be possible to investigate the dynamics of a multiple shaker system with an attached resonant load. Such a study has been initiated and the results are encouraging.

Other studies should be performed to indicate solutions to control problems associated with electrodynamic exciters being employed for shock testing and multiple shaker configurations.

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JET PROPULSION LABORATORY *California Institute of Technology • 4800 Oak Grove Drive, Pasadena, California 91103*

July 1, 1966

Recipients of Jet Propulsion Laboratory
Technical Report No. 32-934

SUBJECT: Errata

Gentlemen:

Please note the following corrections to Jet Propulsion Laboratory Technical Report No. 32-934, "Derivation of the Mathematical Transfer Function of an Electrodynamic Vibration Exciter," May 15, 1966:

Page 2, lines 20, 21.

$$K_g \dot{x} \rightarrow V\text{: voltage, volts}$$

$$K_g \ddot{x} \rightarrow \frac{d}{dt} V = SV\text{: time derivative of voltage}$$

Page 3, line 13.

$$\frac{\ddot{x}_2}{v_d} \text{ or } \frac{S\dot{x}_2}{v_d} \text{ or } \frac{SV_2}{K_g v_d}$$

Page 4, Equation (1).

$$\ddot{x}_2 = \frac{K_g I_3}{C_t}$$

Page 5, equation below Equation (2).

$$\ddot{x}_2 = \frac{v_d (K_g^4 L_f)/C_c}{\Delta_z} \left(\frac{K_g}{C_t} \right)$$

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Page 5, Equation (3).

$$\frac{\ddot{x}_2}{v_d} = \frac{K_g^5 L_f / C_c C_t}{\Delta_z}$$

Page 8, Equation (4).

$$\ddot{x}_2 = \frac{K_g I_3}{C_t}$$

Page 8, Equation (6).

$$\frac{\ddot{x}_2}{v_d} = \frac{K_g^5 L_f / C_c C_t}{\Delta_z}$$

Page 8, four equations at bottom.

$$\frac{\ddot{x}}{v_d} = \frac{K_g^5 L_f}{C_t C_c} \cdot \frac{1}{P/(S^2/K_g^4 L_f L_d L_t)}$$

or

$$\frac{\ddot{x}_2}{v_d} = \frac{K_g^5 L_f}{C_t C_c} \cdot \frac{S^2}{P(K_g^4 L_f L_d L_t)}$$

this becomes

$$\frac{\ddot{x}_2}{v_d} = \frac{S^2 K_g}{(C_t C_c L_d L_t) P}$$

so that

$$\frac{\ddot{x}_2}{v_d} = \frac{S^2 K_g / C_t C_c L_d L_t}{P}$$

Page 9, equation.

$$\ddot{x}/v_d = \frac{S^2 K_g K_t / M_c M_t L_d}{S^5 + S^4\left(\frac{R_d}{L_d}\right) + S^3\left(\frac{K_t}{M_c} + \frac{K_g^2}{L_d M_c} + \frac{K_f}{M_t} + \frac{K_t}{M_t}\right) + S^2 \frac{R_d}{L_d}\left(\frac{K_t}{M_c} + \frac{K_f}{M_t} + \frac{K_t}{M_t}\right) + S \frac{1}{M_c M_t}\left(K_f K_t + \frac{K_g^2 K_f}{L_d} + \frac{K_g^2 K_t}{L_d}\right) + \frac{R_d K_f K_t}{M_c M_t L_d}}$$

Very truly yours,

John Kempton, Manager
Publications Section